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Integrals of periodic motion for classical equations of relativistic string with masses at ends

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Abstract. Boundary equations for the relativistic string with masses at ends are formulated in terms of geometrical invariants of world trajectories of masses at the string ends. In the three-dimensional Minkowski space E_2^1, there are two invariants of that sort, the curvature K and torsion κ. Curvatures of trajectories of the string massive ends are always constant, $K_i = \gamma/m_i (i = 1, 2,)$, whereas torsions $\kappa_i(\tau)$ are the functions of τ and obey a system of differential equations of second order with deviating arguments. For periodic torsions $\kappa_i(\tau + nl) = \kappa(\tau)$, where l is the string length in the plane of parameters τ and $\sigma (0 \leqslant \sigma \leqslant l)$, these equations result in constant of motion.

Introduction

The relativistic string with point masses at ends is the dynamic basis of the string model of hadrons since there is a direct analogy between an open string with masses at its ends and a quark–antiquark pair connected by a tube of the gluon field in quantum chromodynamics [1]. Difficulties of the hadron string model are due to the nonlinear character of boundary conditions, and even at the classical level, the investigation of this system becomes a complicated mathematical problem whose general solution is not yet derived. Therefore, it seems of interest to consider a new mathematical formulation of that problem which would promote the investigation of its dynamics.

The action functional for a relativistic string with masses at its ends results in equations of motion of the string and in boundary conditions that physically represent equations of motion of the two masses interacting through the string. An analogy arises between that system and classical electrodynamics with charges in which the field is described by the Maxwell equations with charges and the dynamics of massive charges interacting with the field is given by Lorentz equations. Wheeler and Feynman [2] considered the action propagating at a distance with a finite velocity, they eliminated the field variables from the equation in electrodynamics and formulated the interaction between charges in terms of retarded and advanced propagation functions when there is no absorption and emission of the electromagnetic field. For a system of relativistic string with masses at its ends, one can also utilize the principle of action at a distance to enable one to find equations of motion in terms of characteristics of the trajectories along which the masses are moving provided the string variables are eliminated. It is clear that owing to the problem being relativistic, it cannot be formulated within the equal-time formalism. In the simplest nonrelativistic limit

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we arrive at a system of two masses coupled by a linearly rising potential [1, 3]. In [4] boundary equations for the relativistic string with masses at ends have been reformulated in the three-dimensional Minkovski space $E_2^1(t, x, y)$ in terms of geometrical invariants of world trajectories of the string ends, their constant curvature K and torsion $\kappa(\tau)$. This pair of invariants determine the trajectory of the string ends with masses up to its position in the space E_2^1 [5]

In this paper, equations for a component of the metric tensor of the world surface of the string at its ends $\dot{x}^2(\tau, \sigma_i)$ are derived from boundary equations of a string with masses at ends; that tensor defines the torsions of boundary curves and for a massless string ($m_i = 0$) it equals zero $\dot{x}^2(\tau, \sigma_i) = 0$. It is shown that these nonlinear equations of second order, when $\dot{x}^2(\tau, \sigma_i)$ are periodic, possess constants of motion that in some cases allow us to reduce the problem of solution to elliptic equations and thus to express $\dot{x}^2(\tau, \sigma_i)$ through elliptic functions in a rational way. In the simplest case of constant $\dot{x}^2(\tau, \sigma_i) = c_i$, we arrive at the well known motion of string ends with masses along helixes and the corresponding world surface of the string turns out to be a helicoid [7]. In the next paper, using some examples with different periods l and $2l$, we will show how the obtained constants of motion allow us to solve the problem of finding the string coordinates. Section 1 is solely a review of previous papers [4, 6]. In section 2, the geometrical approach to boundary equations of relativistic string with masses at ends is formulated in the three-dimensional Minkowski space E_2^1. Section 3 deals with derivation of the constant of motion for boundary equations with periodic torsions of trajectories of the string ends with masses. Section 4 contains some conclusions.

1. Equations of motion and boundary conditions

Classical equations of motion and boundary conditions for a system of two point masses connected by the relativistic string follow from the action function for that system [4, 6]

$$S = -\gamma \int \mathrm{d}\tau \int \mathrm{d}\sigma \sqrt{(\dot{x}\acute{x})^2 - \dot{x}^2\acute{x}^2} - \sum_{i=1}^{2} m_i \int \mathrm{d}\tau \sqrt{\left(\frac{\mathrm{d}x^\mu(\tau_i, \sigma_i(\tau))}{\mathrm{d}\tau}\right)^2}. \tag{1.1}$$

Here the first term is the action of a massless relativistic string; γ is the parameter of tension of the string; m_i are masses of particles at the string ends; $x^\mu(\tau, \sigma)$ are coordinates of the string points in a D-dimensional Minkowski space with metric $(1, -1, -1, \ldots)$; derivatives are denoted by

$$\dot{x}^\mu = \frac{\partial x^\mu(\tau, \sigma)}{\partial \tau} \qquad \acute{x}^\mu = \frac{\partial x^\mu(\tau, \sigma)}{\partial \sigma}$$
$$\frac{\mathrm{d}x^\mu(\tau, \sigma_i(\tau))}{\mathrm{d}\tau} = \dot{x}^\mu(\tau, \sigma_i(\tau)) + \acute{x}^\mu(\tau, \sigma_i(\tau))\dot{\sigma}_i(\tau)$$

where the string endpoints with masses in the plane of parameters τ and σ are described by functions $\sigma_i(\tau)$.

As in the case of a massless string, $m_i = 0$, the action (1.1) is invariant with respect to a nondegenerate change of parameters $\tilde{\tau} = \tilde{\tau}(\tau, \sigma)$ and $\tilde{\sigma} = \tilde{\sigma}(\tau, \sigma)$, which allows us to take the conformally flat metric on the string surface by imposing the conditions of orthonormal gauge

$$\dot{x}^2 + \acute{x}^2 = 0 \qquad \dot{x}\acute{x} = 0. \tag{1.2}$$

The action (1.1) results in the linear equations of motion for the string coordinates [1, 4]

$$\ddot{x}^\mu(\tau, \sigma) - x''^\mu(\tau, \sigma) = 0 \tag{1.3}$$

and the boundary conditions for the ends with masses

$$m_i \frac{\mathrm{d}}{\mathrm{d}\tau}\left[\frac{\dot{x}^\mu(\tau,\sigma_i(\tau)) + \dot{\sigma}(\tau)\acute{x}^\mu(\tau,\sigma_i(\tau))}{\sqrt{\dot{x}^2(\tau,\sigma_i(\tau))(1-\sigma_i{}^2(\tau))}}\right] = (-1)^{i+1}\gamma[\dot{x}^\mu(\tau,\sigma_i(\tau)) + \dot{\sigma}(\tau)\acute{x}^\mu(\tau,\sigma_i(\tau))] \qquad (i=1,2). \tag{1.4}$$

The general solution to equations of motion (1.3) is the vector function

$$x^\mu(\tau,\sigma) = \tfrac{1}{2}[\psi_+^\mu(\tau+\sigma) + \psi_-^\mu(\tau-\sigma)]$$
$$\dot{x}^\mu = \tfrac{1}{2}[\acute{\psi}_+^\mu(\tau+\sigma) + \acute{\psi}_-^\mu(\tau-\sigma)] \qquad \acute{x}^\mu = \tfrac{1}{2}[\acute{\psi}_+^\mu(\tau+\sigma) - \acute{\psi}_-^\mu(\tau-\sigma)] \tag{1.5}$$

where $\acute{\psi}_\pm^\mu(\tau\pm\sigma)$ are derivatives with respect to the arguments.

Inserting it into the gauge conditions (1.2) we obtain the equations

$$\acute{\psi}_+^2(\tau+\sigma) = 0 \qquad \acute{\psi}_-^2(\tau-\sigma) = 0 \tag{1.6}$$

according to which the vectors $\acute{\psi}_+^\mu(\tau+\sigma)$ and $\acute{\psi}_-^\mu(\tau-\sigma)$ should be isotropic. For further consideration, it is convenient to represent them as expansions over a constant basis in the D-dimensional Minkowski space $E_{D_1}^1$ consisting of two isotropic vectors a^μ and $c^\mu(a^\mu a_\mu = 0, c^\mu c_\mu = 0, a^\mu c_\mu = 1)$ and $D-2$ orthonormal space-like vectors $b_k^\mu(r = 1,2,3,\ldots,D-2), b_k^\mu b_{l\mu} = -\delta_{kl}$ orthogonal to vectors a^μ and $c^\mu(a^\mu b_{k\mu} = 0, c^\mu b_{k\mu} = 0)$ [4, 8]. As a result, we obtain the expansion of $\acute{\psi}_\pm^\mu$ over this basis

$$\acute{\psi}_+^\mu(\tau+\sigma) = \frac{A_+(\tau+\sigma)}{\sqrt{\sum_{k=1}^{D-2}\dot{f}_k^2(\tau+\sigma)}}\left[a^\mu + \sum_{k=1}^{D-2} b_k^\mu f_k(\tau+\sigma) + \frac{1}{2}c^\mu\sum_{k=1}^{D-2} f_k^2(\tau+\sigma)\right]$$
$$\acute{\psi}_-^\mu(\tau-\sigma) = \frac{A_-(\tau-\sigma)}{\sqrt{\sum_{k=1}^{D-2}\dot{g}_k^2(\tau-\sigma)}}\left[a^\mu + \sum_{k=1}^{D-2} b_k^\mu g_k(\tau-\sigma) + \frac{1}{2}c^\mu\sum_{k=1}^{D-2} g_k^2(\tau-\sigma)\right]. \tag{1.7}$$

It can easily be verified that $\acute{\psi}_\pm^2 = 0$, condition (1.6) is satisfied and that

$$(\psi_\pm^{\prime\prime\mu}\psi_{\pm\mu}^{\prime\prime}) = \psi_\pm^{\prime\prime 2}(\tau\pm\sigma) = -A_\pm^2(\tau\pm\sigma)$$

where $A_\pm^2(\tau\pm\sigma)$ are two arbitrary functions, such as the functions f_k and g_k. The condition of orthogonal gauge (1.7) does not determine the functions $A_\pm$, and consequently, there is a possibility of fixing them by imposing further gauge conditions since expressions (1.7) are invariant under conformal transformations of the parameters $\tilde{\tau}\pm\tilde{\sigma} = V_\pm(\tau\pm\sigma)$. We fix them by imposing two more gauge conditions

$$[\ddot{x}^\mu(\tau,\sigma) \pm \dot{x}^{\prime\mu}(\tau,\sigma)]^2 = -A^2 = \text{constant} \tag{1.8a}$$

which in terms of the vector functions $\acute{\psi}_\pm^\mu$ mean that the space-like vectors $\psi_\pm^{\prime\prime\mu}(\tau\pm\sigma)$ are modulo constant,

$$\psi_\pm^{\prime\prime 2}(\tau\pm\sigma) = -A_\pm^2(\tau\pm\sigma) = -A^2. \tag{1.8b}$$

In this way, we have fixed the functions $A_\pm(\tau-\sigma)$ now equal to the constant A. At the same time, this condition fixes the values of functions $\sigma_i(\tau)$ (see [4] where it is shown that $\sigma_i(\tau) = \sigma_i = \text{constant}$, therefore, we choose $\sigma_1(\tau) = 0$ and $\sigma_2(\tau) = l$).

Further, we will consider the dynamics of a string with masses at the ends on the plane (x, y), i.e. in the Minkowski space with $D = 2$. In this case, expansion (1.7) contains only one space-like vector b^μ, and expression (1.7) takes the form

$$\acute{\psi}_+^\mu(\tau+\sigma) = \frac{A}{\acute{f}(\tau+\sigma)}[a^\mu + b^\mu f(\tau+\sigma) + (\tfrac{1}{2})c^\mu f^2(\tau+\sigma)]$$
$$\acute{\psi}_-^\mu(\tau-\sigma) = \frac{A}{\acute{g}(\tau-\sigma)}[a^\mu + b^\mu f(\tau-\sigma) + (\tfrac{1}{2})c^\mu g^2(\tau-\sigma)] \tag{1.9}$$

where $\dot{f}(\tau+\sigma), \dot{g}(\tau-\sigma)$ are derivatives with respect to arguments.

2. Boundary equations in terms of invariants of boundary curves

Boundary equations (1.4), when $\sigma_i(\tau) = \text{constant}$, $\dot{x}^\mu(\tau, \sigma_i)$ and $\acute{x}^\mu(\tau, \sigma_i)$ from (1.5) are substituted into them, and representation (1.9) is taken into account, transform into two nonlinear equations for the functions f and g [4]

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\tau} \ln\left[\frac{\acute{g}(\tau)}{\acute{f}(\tau)}\right] + 2\frac{\acute{f}(\tau) + \acute{g}(\tau)}{f(\tau) - g(\tau)} = \frac{\gamma}{m_1}|A|\frac{|f(\tau) - g(\tau)|}{\sqrt{\acute{f}(\tau)\acute{g}(\tau)}} \\ &\frac{\mathrm{d}}{\mathrm{d}\tau} \ln\left[\frac{\acute{g}(\tau - l)}{\acute{f}(\tau + l)}\right] + 2\frac{\acute{f}(\tau + l) + \acute{g}(\tau - l)}{f(\tau + l) - g(\tau - l)} = -\frac{\gamma}{m_2}|A|\frac{|f(\tau + l) - g(\tau - l)|}{\sqrt{\acute{f}(\tau + l)\acute{g}(\tau - l)}} \end{aligned} \tag{2.1}$$

whereas nonzero components of the metric tensor of the string surface $\dot{x}^2(\tau, \sigma) = -\acute{x}^2(\tau, \sigma)$ are expressed via f and g as follows

$$\dot{x}^2(\tau, \sigma) = A^2\frac{[f(\tau + \sigma) - g(\tau - \sigma)]^2}{4\acute{f}(\tau + \sigma)\acute{g}(\tau - \sigma)}. \tag{2.2}$$

As it is known [1], expression (2.2) is the general solution to the Liouville equation

$$\frac{\partial^2 \ln(\dot{x}^2(\tau, \sigma))}{\partial^2\tau} - \frac{\partial^2 \ln(\dot{x}^2(\tau, \sigma))}{\partial^2\sigma} = \frac{A^2}{\dot{x}^2(\tau, \sigma)}$$

which in our case is the Gauss equation for the component of metric tensor of the string minimal surface in the three-dimensional Minkowski space E_2^1 in the gauge (1.2) and (1.8*a*). Geometrically [8, 9], conditions (1.8) mean that the isothermal coordinates (1.2) are at the same time the asymptotic lines on the string world surface.

From (2.2) we obtain the boundary values for the component of metric tensor $\dot{x}^2(\tau, \sigma_i)(\sigma_i = 0, l)$

$$\dot{x}^2(\tau, 0) = A^2\frac{[f(\tau) - g(\tau)]^2}{4\acute{f}(\tau)\acute{g}(\tau)} \qquad \dot{x}^2(\tau, l) = A^2\frac{[f(\tau + l) - g(\tau - l)]^2}{4\acute{f}(\tau + l)\acute{g}(\tau - l)}. \tag{2.3}$$

Now we calculate the curvature $K_i(\tau)$ and torsion $\kappa_i(\tau)$ of boundary curves along which masses $m_i (i = 1, 2)$ are moving. To this end we compare boundary equations (1.4) for $x^\mu(\tau, \sigma_i)$ in the accepted gauge

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\dot{x}_i^\mu(\tau)}{\sqrt{\dot{x}_i^2(\tau)}}\right) = (-1)^{i+1}\frac{\gamma}{m_i}\acute{x}_i^\mu(\tau) \qquad (i = 1, 2) \tag{2.4}$$

with the Frenet–Serret equations [8] for these curves

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\dot{x}_i^\mu(\tau)}{\sqrt{\dot{x}_i^2(\tau)}}\right) = (-1)^{i+1}K_i(\tau)\acute{x}_i^\mu(\tau) \qquad (i = 1, 2) \tag{2.5}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}n_i^\mu(\tau) = \kappa_i(\tau)\acute{x}_i^\mu(\tau) \tag{2.6}$$

where $x_i^\mu(\tau) = x^\mu(\tau, \sigma_i)$, and $n_i^\mu(\tau) = n^\mu(\tau, \sigma_i)$ is a unit space-like vector of the normal that in the chosen basis a^μ, b^μ, c^μ is of the form [4]

$$n^\mu(\tau, \sigma) = \frac{2a^\mu + b^\mu[f(\tau + \sigma) + g(\tau - \sigma)] + c^\mu f(\tau + \sigma)g(\tau - \sigma)}{f(\tau + \sigma) - g(\tau - \sigma)}.$$

By comparing (2.4) with (2.5) we find that the curvatures $K_i(\tau)$ are constant and equal to

$$K_i(\tau) = \gamma/m_i. \tag{2.7}$$

Next, projecting (2.6) onto the vector $\acute{x}_i^{\mu}(\tau)$ and considering that $n^{\mu}(\tau,\sigma)$ is orthogonal to the vectors $\dot{x}^{\mu}(\tau,\sigma)$ and $\acute{x}^{\mu}(\tau,\sigma)$ and $n^2(\tau,\sigma)=-1$, we obtain

$$\kappa_i(\tau)=\frac{(\dot{n}_i\acute{x}_i)}{\acute{x}_i^2(\tau)}=\frac{(n_i\dot{x}_i')}{\dot{x}_i^2(\tau)}=\frac{A}{\dot{x}_i^2(\tau)}\qquad(i=1,3). \tag{2.8}$$

Thus, torsions are determined by $\dot{x}_i^2(\tau)=\dot{x}^2(\tau,\sigma_i)$ and the constant A that is geometrically a nonzero coefficient of the second quadratic form outside the string two-dimensional surface. Indeed, by definition [8, 9],

$$b_{kl}=\left(n\frac{\partial^2 x}{\partial u_k\partial u_l}\right)\qquad\text{where } u_1=\tau, u_2=\sigma$$

$$b_{00}=b_{11}=\tfrac{1}{2}[A_+(\tau+\sigma)-A_-(\tau-\tau)]\qquad b_{01}=b_{10}=\tfrac{1}{2}[A_+(\tau+\sigma)+A_-(\tau-\tau)].$$

Therefore, in our gauge we have $b_{11}=b_{22}=0$ and $b_{12}=b_{21}=A$.

Let us now turn to the boundary equations for functions f and g (2.1) which, in terms of (2.3), allow us to express the differential form of those functions in terms of the constants A, K_i and the component of metric tensor $\dot{x}^2(\tau,\sigma_i)$ on the boundary curves of the string. For this purpose, we write the RHS of equations (2.1), in view of (2.3), in terms of K_i and $\dot{x}^2(\tau,\sigma_i)$ as follows

$$\begin{aligned}\frac{\gamma}{m_1}A\frac{|f(\tau)-g(\tau)|}{\sqrt{\acute{f}(\tau)\acute{g}(\tau)}}&=2K_1\sqrt{\dot{x}^2(\tau,0)}\\ \frac{\gamma}{m_2}A\frac{|f(\tau+l)-g(\tau-l)|}{\sqrt{\acute{f}(\tau+l)\acute{g}(\tau-l)}}&=2K_2\sqrt{\dot{x}^2(\tau,l)}.\end{aligned}\tag{2.9}$$

Then from the first expression of (2.9) we express the difference $f(\tau)-g(\tau)$ via the derivatives $\acute{f}(\tau),\acute{g}(\tau)$ and $\dot{x}^2(\tau,0)$; whereas from the second, the difference $f(\tau+l)-g(\tau-l)$ through the derivatives $\acute{f}(\tau+l),\acute{g}(\tau-l)$ and $\dot{x}^2(\tau,l)$:

$$\begin{aligned}f(\tau)-g(\tau)&=\epsilon[f(\tau)-g(\tau)]\frac{1}{A}\sqrt{\acute{f}(\tau)\acute{g}(\tau)\dot{x}^2(\tau,0)}\\ f(\tau+l)-g(\tau-l)&=\epsilon[f(\tau+l)-g(\tau-l)]\frac{1}{A}\sqrt{\acute{f}(\tau+l)\acute{g}(\tau-l)\dot{x}^2(\tau,l)}\end{aligned}\tag{2.10}$$

where $\epsilon[x]$ is the sign function:

$$\epsilon[x]=\begin{cases}-1 & x<0\\ 1 & x>0\end{cases}.$$

From (2.3) it follows that in view of $\dot{x}^2(\tau,\sigma)>0$ when $m_i\neq 0$, then $\acute{f}(\tau,\sigma)\acute{g}(\tau,\sigma)>0$ throughout. Eliminating the difference $f-g$ from (2.1) by using (2.10), we obtain the boundary equations containing only the derivatives of functions $\acute{f},\acute{g}$ and $\sqrt{\dot{x}^2(\tau,\sigma_i)}$:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\ln\left[\frac{\acute{g}(\tau)}{\acute{f}(\tau)}\right]+\frac{A\epsilon_1}{\sqrt{\dot{x}^2(\tau,0)}}\left(\sqrt{\frac{\acute{f}(\tau)}{\acute{g}(\tau)}}+\sqrt{\frac{\acute{g}(\tau)}{\acute{f}(\tau)}}\right)=2K_1\sqrt{\dot{x}^2(\tau,0)}\tag{2.11a}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\ln\left[\frac{\acute{g}(\tau-l)}{\acute{f}(\tau+l)}\right]+\frac{A\epsilon_2}{\sqrt{\dot{x}^2(\tau,l)}}\left(\sqrt{\frac{\acute{f}(\tau+l)}{\acute{g}(\tau-l)}}+\sqrt{\frac{\acute{g}(\tau-l)}{\acute{f}(\tau+l)}}\right)=-2K_2\sqrt{\dot{x}^2(\tau,l)}\tag{2.11b}$$

where $\epsilon_1=\epsilon[\acute{f}(\tau)\{f(\tau)-g(\tau)\}]$, $\epsilon_2=\epsilon[\acute{f}(\tau+l)\{f(\tau+l)-g(\tau-l)\}]$.

Together with this system of boundary equations, we also consider equalities arising upon the calculation of the logarithmic derivative of (2.3); in this way, with (2.10), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\ln[\acute{g}(\tau)\acute{f}(\tau)] - \frac{A\epsilon_1}{\sqrt{\dot{x}^2(\tau,0)}}\left(\sqrt{\frac{\acute{f}(\tau)}{\acute{g}(\tau)}} - \sqrt{\frac{\acute{g}(\tau)}{\acute{f}(\tau)}}\right) = -\frac{\mathrm{d}}{\mathrm{d}\tau}\ln\dot{x}^2(\tau,0) \tag{2.12a}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\ln[\acute{g}(\tau-l)\acute{f}(\tau+l)] - \frac{A\epsilon_2}{\sqrt{\dot{x}^2(\tau,l)}}\left(\sqrt{\frac{\acute{f}(\tau+l)}{\acute{g}(\tau-l)}} - \sqrt{\frac{\acute{g}(\tau-l)}{\acute{f}(\tau+l)}}\right)$$
$$= -\frac{\mathrm{d}}{\mathrm{d}\tau}\ln\dot{x}^2(\tau,l). \tag{2.12b}$$

The sum and difference of equations (2.11*a*) and (2.11*b*) give two equations

$$\left[2\frac{\mathrm{d}}{\mathrm{d}\tau} + K_1\sqrt{\dot{x}^2(\tau,0)} - \frac{\mathrm{d}}{\mathrm{d}\tau}\ln\sqrt{\dot{x}^2(\tau,0)}\right]\frac{1}{\sqrt{\acute{g}(\tau)}} = \frac{A\epsilon_1}{\sqrt{\dot{x}^2(\tau,0)\acute{f}(\tau)}}$$
$$\left[2\frac{\mathrm{d}}{\mathrm{d}\tau} - K_1\sqrt{\dot{x}^2(\tau,0)} - \frac{\mathrm{d}}{\mathrm{d}\tau}\ln\sqrt{\dot{x}^2(\tau,0)}\right]\frac{1}{\sqrt{\acute{f}(\tau)}} = -\frac{A\epsilon_1}{\sqrt{\dot{x}^2(\tau,0)\acute{g}(\tau)}} \tag{2.13}$$

for the first boundary ($\sigma_1 = 0$). In a similar way, the sum and difference of equations (2.11*b*) and (2.12*b*) of the same systems provide two equations

$$\left[2\frac{\mathrm{d}}{\mathrm{d}\tau} - K_2\sqrt{\dot{x}^2(\tau,l)} - \frac{\mathrm{d}}{\mathrm{d}\tau}\ln\sqrt{\dot{x}^2(\tau,l)}\right]\frac{1}{\sqrt{\acute{g}(\tau-l)}} = \frac{A\epsilon_2}{\sqrt{\dot{x}^2(\tau,l)\acute{f}(\tau+l)}}$$
$$\left[2\frac{\mathrm{d}}{\mathrm{d}\tau} + K_2\sqrt{\dot{x}^2(\tau,l)} - \frac{\mathrm{d}}{\mathrm{d}\tau}\ln\sqrt{\dot{x}^2(\tau,l)}\right]\frac{1}{\sqrt{\acute{f}(\tau+l)}} = -\frac{A\epsilon_1}{\sqrt{\dot{x}^2(\tau,l)\acute{g}(\tau-l)}} \tag{2.14}$$

for the second boundary ($\sigma_2 = l$).

Now let us derive equations separately for the functions $f(\tau)$ and $g(\tau)$ and, respectively, for $f(\tau+l)$ and $g(\tau-l)$. This is achieved by eliminating $1/\sqrt{g'(\tau)}$ from (2.13); thus, for $1/\sqrt{f'(\tau)}$ we obtain

$$\mathrm{D}[f(\tau)] = \mathrm{D}\left[A\int_0^\tau \frac{\mathrm{d}\varsigma}{\sqrt{\dot{x}^2(\varsigma,0)}}\right] + \frac{AK_1}{2}\left(\frac{A}{K_1\dot{x}^2(\tau,0)} - \frac{K_1\dot{x}^2(\tau,0)}{A}\right)$$
$$-2K_1A\frac{\mathrm{d}}{\mathrm{d}\tau}\sqrt{\dot{x}^2(\tau,0)} \tag{2.15}$$

where $\mathrm{D}[y(x)]$ is the Schwartz derivative defined by

$$\mathrm{D}[y(x)] = -2\sqrt{\acute{y}(x)}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{1}{\sqrt{\acute{y}(x)}}\right) = \frac{y'''(x)}{y'(x)} - \frac{3}{2}\left(\frac{y''(x)}{y'(x)}\right)^2. \tag{2.16}$$

Then, removing $1/\sqrt{f'(\tau)}$ from (2.13) we arrive at the equation for $1/\sqrt{g'(\tau)}$

$$\mathrm{D}[g(\tau)] = \mathrm{D}\left[A\int_0^\tau \frac{\mathrm{d}\eta}{\sqrt{\dot{x}^2(\eta,0)}}\right] + \frac{AK_1}{2}\left(\frac{A}{K_1\dot{x}^2(\tau,0)} - \frac{K_1\dot{x}^2(\tau,0)}{A}\right)$$
$$+2K_1A\frac{\mathrm{d}}{\mathrm{d}\tau}\sqrt{\dot{x}^2(\tau,0)}. \tag{2.17}$$

The same procedure for system (2.14) results in the equations for $1/\sqrt{f'(\tau+l)}$ and $1/\sqrt{g'(\tau-l)}$ for the second boundary ($\sigma_2 = l$):

$$\mathrm{D}[f(\tau+l)] = \mathrm{D}\left[A\int^\tau \frac{\mathrm{d}\eta}{\sqrt{\dot{x}^2(\eta,l)}}\right] + \frac{AK_2}{2}\left(\frac{A}{K_2\dot{x}^2(\tau,0)} - \frac{K_2\dot{x}^2(\tau,l)}{A}\right)$$

$$+2K_2A\frac{\mathrm{d}}{\mathrm{d}\tau}\sqrt{\dot{x}^2(\tau,l)} \tag{2.18}$$

$$\mathrm{D}[g(\tau-l)] = \mathrm{D}\left[A\int^{\tau}\frac{\mathrm{d}\eta}{\sqrt{\dot{x}^2(\eta,l)}}\right] + \frac{AK_2}{2}\left(\frac{A}{K_2\dot{x}^2(\tau,l)} - \frac{K_2\dot{x}^2(\tau,l)}{A}\right) -2K_2A\frac{\mathrm{d}}{\mathrm{d}\tau}\sqrt{\dot{x}^2(\tau,l)}. \tag{2.19}$$

From these equations it follows that the functions $f(\tau)$ and $g(\tau)$ are defined by $\dot{x}^2(\tau,0)$ in accordance with (2.15) and (2.17); whereas $f(\tau+l)$ and $g(\tau-l)$, by $\dot{x}^2(\tau,l)$ according to (2.18) and (2.19) since the RHS of these equations contain only $\dot{x}^2(\tau,\sigma_i)$ and constants A, K_i.

For $\dot{x}^2(\tau,0)$ and $\dot{x}^2(\tau,l)$ we can obtain equations that connect them with each other by changing the argument τ in (2.18) to $\tau-l$; and in (2.19) to $\tau+l$, we find that the LHSs of equations (2.15) and (2.18), as well as (2.17) and (2.19) coincide. As a result, we arrive at the two equations

$$\begin{aligned}\mathrm{D}[f(\tau)] &= \mathrm{D}\left[A\int^{\tau}\frac{\mathrm{d}\eta}{\sqrt{\dot{x}^2(\eta,0)}}\right] + \frac{AK_1}{2}\left(\frac{A}{K_1\dot{x}^2(\tau,0)} - \frac{K_1\dot{x}^2(\tau,0)}{A}\right) -2K_1A\frac{\mathrm{d}}{\mathrm{d}\tau}\sqrt{\dot{x}^2(\tau,0)}\\ &= \mathrm{D}\left[A\int^{\tau-l}\frac{\mathrm{d}\eta}{\sqrt{\dot{x}^2(\eta,l)}}\right] + \frac{AK_2}{2}\left(\frac{A}{K_2\dot{x}^2(\tau-l,l)} - \frac{K_2\dot{x}^2(\tau-l,l)}{A}\right) +2K_2A\frac{\mathrm{d}}{\mathrm{d}\tau}\sqrt{\dot{x}^2(\tau-l,l)}\end{aligned} \tag{2.20}$$

$$\begin{aligned}\mathrm{D}[g(\tau)] &= \mathrm{D}\left[A\int^{\tau}\frac{\mathrm{d}\eta}{\sqrt{\dot{x}^2(\eta,0)}}\right] + \frac{AK_1}{2}\left(\frac{A}{K_1\dot{x}^2(\tau,0)} - \frac{K_1\dot{x}^2(\tau,0)}{A}\right) +2K_1A\frac{\mathrm{d}}{\mathrm{d}\tau}\sqrt{\dot{x}^2(\tau,0)}\\ &= \mathrm{D}\left[A\int^{\tau+l}\frac{\mathrm{d}\eta}{\sqrt{\dot{x}^2(\eta,l)}}\right] + \frac{AK_2}{2}\left(\frac{A}{K_2\dot{x}^2(\tau+l,l)} - \frac{K_2\dot{x}^2(\tau+l,l)}{A}\right) -2K_2A\frac{\mathrm{d}}{\mathrm{d}\tau}\sqrt{\dot{x}^2(\tau+l,l)}.\end{aligned} \tag{2.21}$$

The second equalities in (2.20) and (2.21) represent just the connection between $\dot{x}^2(\tau,0)$ and $\dot{x}^2(\tau,l)$.

Further, from (2.15) and (2.17) it follows that the difference of the Schwartz derivatives of the functions $f(\tau)$ and $g(\tau)$ is given by

$$\mathrm{D}[f(\tau)] - \mathrm{D}[g(\tau)] = -4AK_1\frac{\mathrm{d}}{\mathrm{d}\tau}\sqrt{\dot{x}^2(\tau,0)} \tag{2.22}$$

and from (2.18) and (2.19)

$$\mathrm{D}[f(\tau+l)] - \mathrm{D}[g(\tau-l)] = 4AK_2\frac{\mathrm{d}}{\mathrm{d}\tau}\sqrt{\dot{x}^2(\tau,l)}. \tag{2.23}$$

Eliminating $\mathrm{D}[g(\tau)]$ from these equations by changing τ to $\tau+l$ in equation (2.23) and then eliminating $\mathrm{D}[f(\tau)]$ by changing τ to $\tau-l$ in equation (2.23), we obtain the equations

$$\begin{aligned}\mathrm{D}[f(\tau+2l)] - \mathrm{D}[f(\tau)] &= 4A\frac{\mathrm{d}}{\mathrm{d}\tau}\left[K_1\sqrt{\dot{x}^2(\tau,0)} + K_2\sqrt{\dot{x}^2(\tau+l,l)}\right]\\ \mathrm{D}[g(\tau)] - \mathrm{D}[g(\tau-2l)] &= 4A\frac{\mathrm{d}}{\mathrm{d}\tau}\left[K_1\sqrt{\dot{x}^2(\tau,0)} + K_2\sqrt{\dot{x}^2(\tau-l,l)}\right]\end{aligned} \tag{2.24}$$

whose LHSs contain either the function f or g with shifted arguments, whereas the RHSs depend on $\sqrt{\dot{x}^2(\tau,0)}$ and $\sqrt{\dot{x}^2(\tau\pm l,l)}$. These equations give conserved quantities when the difference of the Schwarz derivatives on the LHSs are zero under certain conditions of periodicity to be considered in section 3.

The simplest example of the solution of boundary equations within this approach is the case of constant $\dot{x}^2(\tau,\sigma_i)=\dot{x}_{0i}^2$, i.e. constant torsions of boundary curves $\kappa(\tau,\sigma_i)=\kappa_{0i}$ according to (2.8). It is known [5, 8] that curves in a three-dimensional space with a constant curvature K_i and a constant torsion κ_{0i} are helixes, and the minimal surface within these boundaries is a helicoid. A detailed solution of the corresponding boundary equations is presented in [4]; here we briefly outline the derivation of the solution to that problem in the given approach. From equations (2.22)–(2.24) at $\dot{x}_{0i}^2=$ constant RHSs of these equations are zero and we obtain the equalities

$$\mathrm{D}[f(\tau)]=\mathrm{D}[g(\tau)] \qquad \mathrm{D}[f(\tau+l)]=\mathrm{D}[g(\tau-l)]$$
$$\mathrm{D}[f(\tau+2l)]=\mathrm{D}[f(\tau)] \qquad \mathrm{D}[g(\tau)]=\mathrm{D}[g(\tau+2l)]$$

from which it follows (see the appendix) that the functions entering into the Schwartz derivatives are connected with each other by linear-fractional expressions; in particular, from the first and second equalities it follows that

$$g(\tau)=\frac{a_1f(\tau)+b_1}{c_1f(\tau)+d_1} \qquad g(\tau-l)=\frac{a_2f(\tau+l)+b_2}{c_2f(\tau+l)+d_2} \tag{2.25}$$

where a_i, b_i, c_i, d_i are arbitrary constants such that $a_id_i-b_ic_i=1$. From equations (2.20) and (2.21) we obtain

$$\mathrm{D}[f(\tau)]=\mathrm{D}[g(\tau)]=\frac{AK_1}{2}\left(\frac{A}{K_1\dot{x}_{01}^2}-\frac{K_1\dot{x}_{01}^2}{A}\right)=\frac{AK_2}{2}\left(\frac{A}{K_2\dot{x}_{02}^2}-\frac{K_2\dot{x}_{02}^2}{A}\right). \tag{2.26}$$

Denoting constant quantities equal to each other by ω, which according to [7], is the angular velocity of rotation of a rectilinear string around the centre of rotation, we have the formula

$$K_i\left(\frac{A}{K_i\dot{x}_{0i}^2}-\frac{K_i\dot{x}_{0i}^2}{A}\right)=\omega$$

from which we can express $\dot{x}_{0i}^2$ and the torsion κ_{0i} in terms of ω, A, K_i

$$\kappa_{0i}=\frac{A}{\dot{x}_{0i}^2}=K_i\left(\sqrt{\left(\frac{\omega}{2K_i}\right)^2+1}+\frac{\omega}{2K_i}\right). \tag{2.27}$$

Instead of equations (2.26) that are linear equations of the second order in $1/\sqrt{\acute{f}(\tau)}$ and $1/\sqrt{\acute{g}(\tau)}$, it is easier to determine the functions $f(\tau)$ and $g(\tau)$ from the initial boundary equations (2.1) since they are equations of the second order in $f(\tau)$ and $g(\tau)$ and the ratios of derivatives $g'(\tau)/f'(\tau)$ and $g'(\tau-l)/f'(\tau+l)$ are, according to (2.25), equal to $[c_1f(\tau)+d_1]^{-2}$ and $[c_2f(\tau+l)+d_2]^{-2}$, resp., which reduces equations (2.1) to two equations of the first order whose solution fixes the constants a_i, b_i, c_i, d_i in terms of A, K_i and ω. (In [7], the energy E and angular momentum J have been calculated for such a rotating rectilinear string with a given angular velocity ω and masses m_i at the ends.)

3. Constants of motion for boundary equations of a string with periodic torsions of trajectories of ends

It is a remarkable fact that the system of boundary equations (2.20) and (2.21) possesses conserved quantities when $\dot{x}^2(\tau,\sigma_i)$ are periodic with a period multiple of $l:\dot{x}^2(\tau,\sigma_i)=$

$\dot{x}^2(\tau + nl, \sigma_i), n = 1, 2, 3\ldots$; in this case, the torsions of boundary curves will also be periodic, $\kappa_i(\tau) = \kappa_i(\tau + nl)$.

The RHSs of the above equations depend only on $\dot{x}^2(\tau, \sigma)$, consequently, their LHSs should be periodic with the same period:

$$\mathrm{D}[f(\tau + nl)] = \mathrm{D}[f(\tau)] \qquad \mathrm{D}[g(\tau + nl)] = \mathrm{D}[g(\tau)]. \tag{3.1}$$

In view of the property of the Schwartz derivative (see the appendix) we have

$$\begin{aligned} f(\tau + nl) &= \frac{a_1 f(\tau) + b_1}{c_1 f(\tau) + d_1} = T_1 f(\tau) \\ g(\tau + nl) &= \frac{a_2 g(\tau) + b_2}{c_2 g(\tau) + d_2} = T_2 g(\tau). \end{aligned} \tag{3.2}$$

We will prove that these two linear-fractional transformations are to be equal: $T_1 = T_2$. To this end, using (3.2) and (2.3), we write the condition of periodicity for $\dot{x}^2(\tau, \sigma_i)$

$$\begin{aligned} \dot{x}^2(\tau, 0) &= \frac{A^2[f(\tau) - g(\tau)]^2}{4\acute{f}(\tau)\acute{g}(\tau)} = \dot{x}^2(\tau + nl, 0) = \frac{A^2[T_1 f(\tau) - T_2 g(\tau)]^2}{4(T_1 f(\tau))'(T_2 f(\tau))'} \\ \dot{x}^2(\tau, l) &= \frac{A^2[f(\tau + l) - g(\tau - l)]^2}{4\acute{f}(\tau + l)\acute{g}(\tau - l)} = \dot{x}^2(\tau + nl, l) \\ &= \frac{A^2[T_1 f(\tau + l) - T_2 g(\tau - l)]^2}{4(T_1 f(\tau + l))'(T_2 f(\tau - l))'}. \end{aligned} \tag{3.3}$$

Since the derivatives of the linear-fractional function are given by the expressions

$$(T_1 f(\tau))' = \frac{f'(\tau)}{[c_1 f(\tau) + d_1]^2} \qquad (T_2 g(\tau))' = \frac{g'(\tau)}{[c_2 g(\tau) + d_2]^2}$$

the denominators in (3.3) coincide, and the numerators obey the equality

$$[f(\tau) - g(\tau)] = (a_1 f(\tau) + b_1)(c_2 g(\tau) + d_2) - (c_1 f(\tau) + d_1)(a_2 g(\tau) + b_2)$$

and the same equality follows from the second equation of (3.3) but with shifted arguments of $f(\tau + l)$ and $g(\tau - l)$. These equalities, provided that $a_i d_i - b_i c_i = 1$, hold valid under the condition

$$a_1 = a_2 = a \qquad b_1 = b_2 = b \qquad c_1 = c_2 = c \qquad d_1 = d_2 = d.$$

Thus, the periodicity condition (3.3) results in that f and g are transformed as follows

$$f(\tau + nl) = T f(\tau) \qquad g(\tau + nl) = T g(\tau) \qquad \text{where } T f(\tau) = \frac{a f(\tau) + b}{c f(\tau) + d}. \tag{3.4}$$

Now we can consider each of the periods $l, 2l, \ldots, nl$ separately and consequences that follow from equations (2.24) in these cases.

For the period l, $\dot{x}^2(\tau + l, \sigma_i) = \dot{x}^2(\tau, \sigma_i)$ from equation (3.4) it follows that $f(\tau + l) = T f(\tau)$ and $g(\tau - l) = T^{-1} g(\tau)$, where T^{-1} is the inverse linear-fractional transformation, and

$$f(\tau + 2l) = T(T f(\tau)) = \frac{(a^2 + cb) f(\tau) + b(a + d)}{c(a + d) f(\tau) + d^2 + cb} = \frac{[a - (a + d)^{-1}] f(\tau) + b}{c f(\tau) + d - (a + d)^{-1}}$$

$$g(\tau - 2l) = T^{-1}(T^{-1} g(\tau)) = \frac{(d^2 + cb) g(\tau) - b(a + d)}{-c(a + d) g(\tau) + a^2 + cb} = \frac{[d - (a + d)^{-1}] g(\tau) - b}{-c g(\tau) + a - (a + d)^{-1}}$$

are also linear-fractional transformations with the determinant equal to unity when $ad - bc = 1$. Then, the LHS of equations (2.24) are zero because

$$\mathrm{D}[f(\tau + 2l)] = \mathrm{D}[f(\tau)] \qquad \mathrm{D}[g(\tau)] = \mathrm{D}[g(\tau - 2l)]$$

and from (2.24) we obtain the conserved quantity for the motion with $\dot{x}^2(\tau,\sigma_i) = \dot{x}^2(\tau+l,\sigma_i)$

$$K_1\sqrt{\dot{x}^2(\tau,0)} + K_2\sqrt{\dot{x}^2(\tau,l)} = h_1 \tag{3.5}$$

where h_1 is the constant of integration.

For equal masses at the string ends $m_1 = m_2, K_1 = K_2 = \gamma/m$, and if we put $f(\tau+l) = g(\tau)$and$g(\tau-l) = f(\tau)$, the equality $\dot{x}^2(\tau,0) = \dot{x}^2(\tau,l)$ is fulfilled and the second boundary (2.1) turns into the first one. Then equation (3.5) results in constant $\dot{x}^2(\tau,\sigma_i)$ because

$$K\sqrt{\dot{x}^2(\tau,0)} = K\sqrt{\dot{x}^2(\tau,l)} = h_1/2.$$

Now, let us consider the case with period $2l$: $\dot{x}^2(\tau+2l,\sigma_i) = \dot{x}^2(\tau,\sigma_i)$. According to (3.4), $f(\tau+2l) = Tf(\tau)$ and $g(\tau-2l) = T^{-1}g(\tau)$, therefore, $\mathrm{D}[f(\tau+2l)] = \mathrm{D}[f(\tau)]$ and $\mathrm{D}[g(\tau-2l)] = \mathrm{D}[g(\tau)]$, then the LHSs of equations (2.24) again turn out to be zero; upon integration we obtain

$$K_1\sqrt{\dot{x}^2(\tau,0)} + K_2\sqrt{\dot{x}^2(\tau\pm l,l)} = h_2. \tag{3.6}$$

This constant of motion for the period $2l$ differs from (3.5) by the argument in the second term shifted by l. Therefore, when masses are equal, $K_1 = K_2$, and the special case, $\dot{x}^2(\tau,0) = \dot{x}^2(\tau,l) = \dot{x}^2(\tau)$, $f(\tau+l) = g(\tau)$, is considered, we do not obtain constant $\dot{x}^2(\tau,\sigma_i)$ since in this special case (3.6) results in the expression

$$K\left[\sqrt{\dot{x}^2(\tau)} + \sqrt{\dot{x}^2(\tau\pm l)}\right] = h_2 \tag{3.7}$$

which is fulfilled not only for constant $\dot{x}^2(\tau)$. The derivation of $\dot{x}^2(\tau,\sigma)$ and solution of the whole problem for the period $2l$ will be done in the next paper.

The case with period $3l$ is more complicated. For this period, from the first of equations (2.24), by shifting the argument τ by l and then by $2l$, we obtain the system of three equations:

$$\begin{aligned}
\mathrm{D}[f(\tau+2l)] - \mathrm{D}[f(\tau)] &= 4A\frac{\mathrm{d}}{\mathrm{d}\tau}\left[K_1\sqrt{\dot{x}^2(\tau,0)} + K_2\sqrt{\dot{x}^2(\tau+l,l)}\right]\\
\mathrm{D}[f(\tau+3l)] - \mathrm{D}[f(\tau+l)] &= 4A\frac{\mathrm{d}}{\mathrm{d}\tau}\left[K_1\sqrt{\dot{x}^2(\tau+l,0)} + K_2\sqrt{\dot{x}^2(\tau+2l,l)}\right]\\
\mathrm{D}[f(\tau+4l)] - \mathrm{D}[f(\tau+2l)] &= 4A\frac{\mathrm{d}}{\mathrm{d}\tau}\left[K_1\sqrt{\dot{x}^2(\tau+2l,0)} + K_2\sqrt{\dot{x}^2(\tau+3l,l)}\right].
\end{aligned} \tag{3.8}$$

Summing these equalities and considering that $\mathrm{D}[f(\tau+3l)] = \mathrm{D}[f(\tau)]$, $\mathrm{D}[f(\tau+4l)] = \mathrm{D}[f(\tau+l)]$ and $\dot{x}^2(\tau+3l,l) = \dot{x}^2(\tau,l)$, we obtain

$$\begin{aligned}
0 = 4A\frac{\mathrm{d}}{\mathrm{d}\tau}&\left\{K_1\left(\sqrt{\dot{x}^2(\tau,0)} + \sqrt{\dot{x}^2(\tau+l,0)} + \sqrt{\dot{x}^2(\tau+2l,0)}\right)\right.\\
&\left.+ K_2\left(\sqrt{\dot{x}^2(\tau,l)} + \sqrt{\dot{x}^2(\tau+l,l)} + \sqrt{\dot{x}^2(\tau+2l,l)}\right)\right\}
\end{aligned}$$

and upon integration we have

$$\sum_{m=0}^{2}\left[K_1\sqrt{\dot{x}^2(\tau+ml,0)} + K_2\sqrt{\dot{x}^2(\tau+ml,l)}\right] = h_3. \tag{3.9}$$

In the same way, from the second of equations (2.24), by shifting the argument τ by $-l$, and then by $-2l$, we obtain three equations, the sum of which gives

$$4A\sum_{m=0}^{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\left[K_1\sqrt{\dot{x}^2(\tau-ml,0)} + K_2\sqrt{\dot{x}^2(\tau-ml,l)}\right] = 0. \tag{3.10}$$

This expression coincides with (3.9) when τ is changed to $\tau + 2l$.

From these examples it is not difficult to deduce the general expression for a conserved quantity for period nl that is different for even and odd n.

For even $n = 2r (r = 1, 2, \ldots)$, it is necessary, upon adding $2ml$ to the argument τ in equation (2.24), to sum up the obtained expressions over m from zero to $r-1$, which gives

$$\sum_{m=0}^{r-1} \mathrm{D}[f(\tau + 2(1+m)l)] - \sum_{m=0}^{r-1} \mathrm{D}[f(\tau + 2ml)]$$
$$= 4A \sum_{m=0}^{r-1} \frac{\mathrm{d}}{\mathrm{d}\tau} \left\{ K_1 \sqrt{\dot{x}^2(\tau + 2ml, 0)} + K_2 \sqrt{x^2(\tau + (1+2m)l, l)} \right\}.$$

The LHS of this equation equals zero since under the change $1 + m = m'$ in the first sum we have

$$\sum_{m'=1}^{r} \mathrm{D}[f(\tau + 2m'l)] - \sum_{m=0}^{r-1} \mathrm{D}[f(\tau + 2ml)] = \mathrm{D}[f(\tau + 2rl)] - \mathrm{D}[f(\tau)] = 0$$

and hence the constant quantity is

$$\sum_{m=0}^{r-1} \left\{ K_1 \sqrt{\dot{x}^2(\tau + 2ml, 0)} + K_2 \sqrt{x^2(\tau + l + 2ml, l)} \right\} = h_{2r}. \tag{3.11}$$

When $r = 1$, from (3.11) we obtain (3.6) with period $2l$.

For odd $n = 2r + 1 (r = 0, 1, 2, \ldots)$, it is necessary, adding ml to the argument in (2.24), to sum up the equations over m from zero to $2r$, then

$$\sum_{m=0}^{2r} \mathrm{D}[f(\tau + 2l + ml)] - \sum_{m=0}^{2r} \mathrm{D}[f(\tau + ml)]$$
$$= 4A \sum_{m=0}^{2r} \frac{\mathrm{d}}{\mathrm{d}\tau} \left\{ K_1 \sqrt{\dot{x}^2(\tau + ml, 0)} + K_2 \sqrt{x^2(\tau + l + ml, l)} \right\}. \tag{3.12}$$

Again, the LHS of equation (3.12) is zero since setting $2 + m = m'$ in the first sum and considering that $(1 + 2k)l$ is a period, we obtain

$$\sum_{m'=2}^{2+2r} \mathrm{D}[f(\tau + m'l)] - \sum_{m=0}^{2r} \mathrm{D}[f(\tau + ml)] = \mathrm{D}[f(\tau + (1+2r)l)]$$
$$-\mathrm{D}[f(\tau + 2(1+r)l)] - \mathrm{D}[f(\tau)] - \mathrm{D}[f(\tau + l)] = 0.$$

Consequently, in this case the quantity

$$\sum_{m=0}^{2r} \left\{ K_1 \sqrt{\dot{x}^2(\tau + ml, 0)} + K_2 \sqrt{x^2(\tau + ml, l)} \right\} = h_{2r+1} \tag{3.13}$$

is constant. In (3.13) we considered that the last term in the sum of the second term in (3.12) equals $\dot{x}^2(\tau + l + 2rl, l) = \dot{x}^2(\tau, l)$. When $r = 0$ and $r = 1$, we obtain (3.5) and (3.6).

So, (3.11) and (3.13) are constants of motion of the boundary equations of a relativistic string with masses at ends when masses are moving along the curves with periodic torsion $\kappa_i(\tau + nl) = \kappa_i(\tau)$ and constant curvature $K_i = \gamma/m_i$. The curves with constant curvatures in the Euclidean geometry are called the Bertrand curves [9]; in our case, when $K_1 = K_2 (m_1 = m_2)$, two boundary curves along which the masses m_i are moving are two conjugate Bertrand curves, i.e. the centre of curvature of one curve is always on the other curve.

4. Conclusion

The constants of motion, (3.11) and (3.13), obtained can be geometrically interpreted as follows. Since the length of a curve L_i between points τ_2 and τ_1 is given by the expression

$$L_i(\tau_2, \tau_1) = \int_{\tau_2}^{\tau_1} \sqrt{\dot{x}^2(\tau, \sigma_i)}\, \mathrm{d}\tau \tag{4.1}$$

then, integrating (3.11), (3.13) in the interval $[\tau_2, \tau_1]$ and expressing the curvature K_i through the curvature radius R_i: $R_i = 1/K_i$, we obtain

$$\sum_{m=0}^{r-1} \left[\frac{1}{R_1} L_1(\tau_1 + 2ml, \tau_2 + 2ml) + \frac{1}{R_2} L_2(\tau_1 + 2ml, \tau_2 + 2ml) \right] = h_{2r}(\tau_1 - \tau_2) \tag{4.2}$$

$$\sum_{m=0}^{2r} \left[\frac{1}{R_1} L_1(\tau_1 + ml, \tau_2 + ml) + \frac{1}{R_2} L_2(\tau_1 + ml, \tau_2 + ml) \right] = h_{2r+1}(\tau_1 - \tau_2). \tag{4.3}$$

From these expressions it is seen that sums of the curves divided by constant radii R_i of their curvatures grow linearly with the parameter τ as though their element of length were a constant $\sqrt{\dot{x}_{0i}^2}$. Consequently, we can set the constant h_{2r} in (3.11) to be equal to

$$h_{2r} = r \left(K_1 \sqrt{\dot{x}_{01}^2} + K_2 \sqrt{\dot{x}_{02}^2} \right)$$

whereas the constant h_{2r+1} in (3.13) to be equal to

$$h_{2r+1} = (2r+1) \left(K_1 \sqrt{\dot{x}_{01}^2} + K_2 \sqrt{\dot{x}_{02}^2} \right).$$

In particular, for the period l in (3.5) $r = 0$, and for the period $2l$ in (3.6) $r = 1$, therefore,

$$h_1 = h_2 = K_1 \sqrt{\dot{x}_{01}^2} + K_2 \sqrt{\dot{x}_{02}^2}.$$

It has been mentioned that the constants $\sqrt{\dot{x}^2(\tau, \sigma_i)}$ arise only when masses m_i move along helixes and thus the sum of lengths along which m_i passes in the case of periodic motion during the intervals of τ multiple of l up to nl is equal to the length passed by the same point m_i as if it was moving along a helix with constant $\dot{x}_{01}^2$.

In a subsequent paper, we will analyse boundary equations (2.1) for periodic $\dot{x}^2(\tau, \sigma_i)$ to show how the integrals of motion we have here derived can be applied to find the world surface of a string when $\dot{x}^2(\tau, \sigma_i)$ has periods l and $2l$. These solutions are expressed through elliptic functions and describe motion of the relativistic string that is more complicated than rotation of the string as a finite straight line, therefore, the string world surfaces for both periods l and $2l$ do not belong to the class of ruled surfaces. A solution of that sort describes transverse excitations of the string and radial motions of masses. The paper deals with a string in three-dimensional space E_2^1, but in 4-dimensional space E_3^1, where the amount of unknown functions increases up to four, since according to (1.7) and (1.8*b*), we have

$$\acute{\psi}_+^\mu(\tau + \sigma) = \frac{A}{\sqrt{\acute{f}_1^2(\tau + \sigma) + \acute{f}_2^2(\tau + \sigma)}} \left[a^\mu + b_1^\mu f_1(\tau + \sigma) + b_2^\mu f_2(\tau + \sigma) \right.$$
$$\left. + c^\mu \frac{f_1^2(\tau + \sigma) + f_2^2(\tau + \sigma)}{2} \right]$$

$$\acute{\psi}_-^\mu(\tau - \sigma) = \frac{A}{\sqrt{\acute{g}_1^2(\tau - \sigma) + \acute{g}_2^2(\tau - \sigma)}} \left[a^\mu + b_1^\mu g_1(\tau - \sigma) + b_2^\mu g_2(\tau - \sigma) \right.$$

$$+c^{\mu}\frac{g_1^2(\tau-\sigma)+g_2^2(\tau-\sigma)}{2}\Bigg]$$

and instead of the component of the metric tensor $\dot{x}^2(\tau,\sigma)$ in (2.2) we obtain

$$\dot{x}^2(\tau,\sigma)=\frac{A^2}{4}\frac{[f_1(\tau+\sigma)-g_1(\tau-\sigma)]^2+[f_2(\tau+\sigma)-g_2(\tau-\sigma)]^2}{\sqrt{[\acute{f}_1^2(\tau+\sigma)+\acute{f}_2^2(\tau+\sigma)][\acute{g}_1^2(\tau-\sigma)+\acute{g}_2^2(\tau-\sigma)]}}.$$

Hence, it is seen that this expression is not invariant under linear-fractional substitutions (3.2) for all the functions f_i and g_i, in contrast to three-dimensional space (2.2). Therefore, when the argument τ is shifted by a period of nl, formulae (3.2) are invalid.

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Appendix

An important property of the Schwartz derivative $\mathrm{D}[f(\tau)]$ defined by

$$\mathrm{D}[f(\tau)]=\frac{f'''(\tau)}{f'(\tau)}-\frac{3}{2}\left(\frac{f''(\tau)}{f'(\tau)}\right)^2=-2\sqrt{f'(\tau)}\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\left(\frac{1}{\sqrt{f'(\tau)}}\right) \qquad \text{(A.1)}$$

is that it is invariant under linear-fractional transformations of the function $f(\tau)$

$$f(\tau)\to\frac{af(\tau)+b}{cf(\tau)+d} \qquad ad-bc=1$$

which is easily proved by using the second form for $\mathrm{D}[f(\tau)]$ given in (A.1) and considering that $f'(\tau)\to f'(\tau)[cf(\tau)+d]^{-2}$.

One more important property of these derivatives consists of the fact that from the equality of the Schwartz derivatives of two functions $f(\tau)$ and $g(\tau)$ it follows that these functions are connected with each other via a linear-fractional transformation. Indeed, from $\mathrm{D}[f(\tau)]=\mathrm{D}[g(\tau)]$ it follows that

$$\sqrt{f'(\tau)}\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\left(\frac{1}{\sqrt{f'(\tau)}}\right)=\sqrt{g'(\tau)}\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\left(\frac{1}{\sqrt{g'(\tau)}}\right)$$

or, assuming that $f'(\tau), g'(\tau)\neq 0$, we have

$$0=\frac{1}{\sqrt{g'(\tau)}}\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\left(\frac{1}{\sqrt{f'(\tau)}}\right)-\frac{1}{\sqrt{f'(\tau)}}\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\left(\frac{1}{\sqrt{g'(\tau)}}\right)$$
$$=\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\frac{1}{\sqrt{g'(\tau)}}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{\sqrt{f'(\tau)}}\right)-\frac{1}{\sqrt{f'(\tau)}}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{\sqrt{g'(\tau)}}\right)\right].$$

After integration, we obtain

$$\frac{1}{\sqrt{g'(\tau)}}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{\sqrt{f'(\tau)}}\right)-\frac{1}{\sqrt{f'(\tau)}}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{\sqrt{g'(\tau)}}\right)=-c. \qquad \text{(A.2)}$$

Then multiplying (A.2) by $f'(\tau)$ we arrive at the total derivative

$$\frac{f'(\tau)}{\sqrt{g'(\tau)}}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{\sqrt{f'(\tau)}}\right)-\sqrt{f'(\tau)}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{\sqrt{g'(\tau)}}\right)=-\frac{\mathrm{d}}{\mathrm{d}\tau}\sqrt{\frac{f'(\tau)}{g'(\tau)}}=-cf'(\tau).$$

As a result, $g'(\tau) = f'(\tau)[cf(\tau) + d]^{-2}$, and thus,

$$g(\tau) = \frac{af(\tau) + b}{cf(\tau) + d} \qquad \text{where } ad - bc = 1.$$

Note added in proof. Recently similar topics have been considered in a paper by Capovilla and Guven (Capovilla R and Guven J 1997 Extended objects with edges *Phys. Rev.* D **55** 2388).

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